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The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and the deformation of W_N algebra

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Abstract

After reviewing the recent results on the Drinfeld realization of the face-type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ by the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, we investigate a fusion of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. The basic generating functions $\Lambda_j(z)$ $(1 \leqslant j \leqslant N-1)$ of the deformed W_N algebra are derived explicitly.

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1. Introduction

In recent papers [1–3], we showed that the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ provides the Drinfeld realization of the face-type elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ [4] tensored by a Heisenberg algebra. Based on this fact, we defined the $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ counterparts of the intertwining operators of the $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ modules and obtained their free-field realization in the level-one representation. The resultant vertex operators, called the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, are identified with the vertex operators of the $\widehat{\mathfrak{sl}}_N$-type RSOS model in the algebraic analysis formulation [5]. In general, we expect that the elliptic algebra $U_{q,p}(\mathfrak{g})$ with $\mathfrak{g}$ being an affine Lie algebra provides the Drinfeld realization for the elliptic quantum group $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ and enables us to perform an algebraic analysis of the $\mathfrak{g}$-type RSOS model.

On the other hand, the $\widehat{\mathfrak{sl}}_N$ RSOS model is known as an off-critical deformation of the W_N minimal model [6]. In this relation, it is remarkable that the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ in the $c = 1$ representation coincides with the algebra of the screening currents of the deformed W_N algebra [7–9]. In general, we expect that the elliptic algebra $U_{q,p}(\mathfrak{g})$ provides an algebra of screening currents of the deformation of the coset CFT associated with $(\mathfrak{g})_c \oplus (\mathfrak{g})_{r-c-2}/(\mathfrak{g})_{r-2}$ [1, 2], which corresponds to the $c \times c$ fusion RSOS model of type $\mathfrak{g}$.

The purpose of this paper is to continue to discuss an explicit relation among the elliptic algebra $U_{q,p}(\mathfrak{g})$, the $\mathfrak{g}$-type RSOS model and the deformation of $W(\bar{\mathfrak{g}})$ algebra in the case $\mathfrak{g} = \widehat{\mathfrak{sl}}_N$. We here investigate a fusion of the type-II vertex operator of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and its dual, and show that the generating functions of the deformed W_N algebra can be extracted from it. The idea of fusion of the vertex operators was used in [10, 11] to derive the generating function of the deformed Virasoro algebra (corresponding to the $\phi_{1,3}$ perturbation) from the ABF model in regime III, in [12] for the deformed W_N algebra with the central charge $c_N = (N-1)\left(1 - \frac{N(N+1)}{r(r-1)}\right)$ at special point $r = N+2$ (the $\mathbb{Z}_N$ parafermion point) from the ABF model in regime II, and in [13] for the deformed Virasoro algebra (corresponding to the $\phi_{1,2}$ perturbation) from the dilute A_L model.

This paper is organized as follows. In the next section, we briefly review the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as the Drinfeld realization of the elliptic quantum group $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ according to [3]. In section 3, we give a summary of the results on the free-field realization of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. In section 4, we discuss a fusion of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and derive the basic generators of the deformed W_N algebra.

Throughout this paper, we use the following symbols: $p = q^{2r}$, $p^* = pq^{-2c} = q^{2r^*}$ ($r^* = r - c$; $r, r^* \in \mathbb{R}_{>0}$),

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$\Theta_p(z) = (z, p)_\infty (pz^{-1}; p)_\infty (p; p)_\infty$$

$$\{z\} = (z; p, q^{2N})_\infty \qquad \{z\}^* = \{z\}|_{p \to p^*}$$

$$(z; t_1, \ldots, t_k)_\infty = \prod_{n_1,\ldots,n_k \geqq 0} \left(1 - z t_1^{n_1} \cdots t_k^{n_k}\right).$$

We also use the Jacobi theta functions

$$[v] = q^{\frac{v^2}{r} - v} \frac{\Theta_p(q^{2v})}{(p; p)_\infty^3} \qquad [v]^* = q^{\frac{v^2}{r^*} - v} \frac{\Theta_{p^*}(q^{2v})}{(p^*; p^*)_\infty^3}$$

which satisfy $[-v] = -[v]$ and the quasi-periodicity property

$$[v + r] = -[v] \qquad [v + r\tau] = -\mathrm{e}^{-\pi \mathrm{i}\tau - \frac{2\pi \mathrm{i} v}{r}} [v].$$

We take the normalization of the theta function to be

$$\oint_{C_0} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \frac{1}{[-v]} = 1 \qquad \oint_{C_0} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \frac{1}{[-v]^*} = \left. \frac{[v]}{[v]^*} \right|_{v \to 0}$$

where C_0 is a simple closed curve in the v-plane encircling $v = 0$ anticlockwise.

2. The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

2.1. Definition

Definition 2.1 (elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$). *We define the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ to be the associative algebra of the currents $E_j(v)$, $F_j(v)$ $(1 \leqslant j \leqslant N-1)$ and $K_j(v)$ $(1 \leqslant j \leqslant N)$ satisfying the following relations:*

$$E_i(v_1) E_j(v_2) = \frac{\left[v_1 - v_2 + \frac{A_{ij}}{2}\right]^*}{\left[v_1 - v_2 - \frac{A_{ij}}{2}\right]^*} E_j(v_2) E_i(v_1) \tag{2.1}$$

$$F_i(v_1) F_j(v_2) = \frac{\left[v_1 - v_2 - \frac{A_{ij}}{2}\right]}{\left[v_1 - v_2 + \frac{A_{ij}}{2}\right]} F_j(v_2) F_i(v_1) \tag{2.2}$$

$$[E_i(v_1), F_j(v_2)] = \frac{\delta_{i,j}}{q-q^{-1}}\left(\delta(q^{-c}z_1/z_2)H_j^+\left(v_2+\frac{c}{4}\right)-\delta(q^c z_1/z_2)H_j^-\left(v_2-\frac{c}{4}\right)\right) \tag{2.3}$$

$$H_j^{\pm}\left(v \mp \frac{1}{2}\left(r-\frac{c}{2}\right)\right) = \kappa K_j\left(v+\frac{N-j}{2}\right)K_{j+1}\left(v+\frac{N-j}{2}\right)^{-1} \tag{2.4}$$

$$K_j(v_1)K_j(v_2) = \rho(v_1-v_2)K_j(v_2)K_j(v_1) \tag{2.5}$$

$$K_{j_1}(v_1)K_{j_2}(v_2) = \rho(v_1-v_2)\frac{[v_1-v_2-1]^*[v_1-v_2]}{[v_1-v_2]^*[v_1-v_2-1]}K_{j_2}(v_2)K_{j_1}(v_1) \quad (1 \leqq j_1 < j_2 \leqq N) \tag{2.6}$$

$$K_j(v_1)E_j(v_2) = \frac{\left[v_1-v_2+\frac{j+r^*-N}{2}\right]^*}{\left[v_1-v_2+\frac{j+r^*-N}{2}-1\right]^*}E_j(v_2)K_j(v_1) \tag{2.7}$$

$$K_{j+1}(v_1)E_j(v_2) = \frac{\left[v_1-v_2+\frac{j+r^*-N}{2}\right]^*}{\left[v_1-v_2+\frac{j+r^*-N}{2}+1\right]^*}E_j(v_2)K_{j+1}(v_1) \tag{2.8}$$

$$K_{j_1}(v_1)E_{j_2}(v_2) = E_{j_2}(v_2)K_{j_1}(v_1) \qquad (j_1 \neq j_2, j_2+1) \tag{2.9}$$

$$K_j(v_1)F_j(v_2) = \frac{\left[v_1-v_2+\frac{j+r-N}{2}-1\right]}{\left[v_1-v_2+\frac{j+r-N}{2}\right]}F_j(v_2)K_j(v_1) \tag{2.10}$$

$$K_{j+1}(v_1)F_j(v_2) = \frac{\left[v_1-v_2+\frac{j+r-N}{2}+1\right]}{\left[v_1-v_2+\frac{j+r-N}{2}\right]}F_j(v_2)K_{j+1}(v_1) \tag{2.11}$$

$$K_{j_1}(v_1)F_{j_2}(v_2) = F_{j_2}(v_2)K_{j_1}(v_1) \qquad (j_1 \neq j_2, j_2+1) \tag{2.12}$$

$$\begin{aligned} z_1^{-\frac{1}{r^*}}\frac{(p^*q^2z_2/z_1;p^*)_\infty}{(p^*q^{-2}z_2/z_1;p^*)_\infty}&\left\{(z_2/z)^{\frac{1}{r^*}}\frac{(p^*q^{-1}z/z_1;p^*)_\infty(p^*q^{-1}z/z_2;p^*)_\infty}{(p^*qz/z_1;p^*)_\infty(p^*qz/z_2;p^*)_\infty}E_i(v_1)E_i(v_2)E_j(v)\right.\\ &-[2]_q\frac{(p^*q^{-1}z/z_1;p^*)_\infty(p^*q^{-1}z_2/z;p^*)_\infty}{(p^*qz/z_1;p^*)_\infty(p^*qz_2/z;p^*)_\infty}E_i(v_1)E_j(v)E_i(v_2)\\ &\left.+(z/z_1)^{\frac{1}{r^*}}\frac{(p^*q^{-1}z_1/z;p^*)_\infty(p^*q^{-1}z_2/z;p^*)_\infty}{(p^*qz_1/z;p^*)_\infty(p^*qz_2/z;p^*)_\infty}E_j(v)E_i(v_1)E_i(v_2)\right\}\\ &+(z_1\leftrightarrow z_2)=0 \end{aligned} \tag{2.13}$$

$$\begin{aligned} z_1^{\frac{1}{r}}\frac{(pq^{-2}z_2/z_1;p)_\infty}{(pq^2z_2/z_1;p)_\infty}&\left\{(z/z_2)^{\frac{1}{r}}\frac{(pqz/z_1;p)_\infty(pqz/z_2;p)_\infty}{(pq^{-1}z/z_1;p)_\infty(pq^{-1}z/z_2;p)_\infty}F_i(v_1)F_i(v_2)F_j(v)\right.\\ &-[2]_q\frac{(pqz/z_1;p)_\infty(pqz_2/z;p)_\infty}{(pq^{-1}z/z_1;p)_\infty(pq^{-1}z_2/z;p)_\infty}F_i(v_1)F_j(v)F_i(v_2)\\ &\left.+(z_1/z)^{\frac{1}{r}}\frac{(pqz_1/z;p)_\infty(pqz_2/z;p)_\infty}{(pq^{-1}z_1/z;p)_\infty(pq^{-1}z_2/z;p)_\infty}F_j(v)F_i(v_1)F_i(v_2)\right\}\\ &+(z_1\leftrightarrow z_2)=0 \qquad (|i-j|=1). \end{aligned} \tag{2.14}$$

Here $A = (A_{jk})$ is the Cartan matrix of $\mathfrak{sl}_N$. The constant κ and the functions $\rho(v)$ are given by

$$\kappa = \frac{(p;p)_\infty (p^*q^2;p^*)_\infty}{(p^*;p^*)_\infty (pq^2;p)_\infty} \tag{2.15}$$

$$\rho(v) = \frac{\rho^{+*}(v)}{\rho^+(v)} \tag{2.16}$$

$$\rho^+(v) = q^{\frac{N-1}{N}} z^{\frac{N-1}{rN}} \frac{\{pq^2z\}\{pq^{2N-2}z\}\{1/z\}\{q^{2N}/z\}}{\{pz\}\{pq^{2N}z\}\{q^2/z\}\{q^{2N-2}/z\}} \qquad \rho^{+*}(v) = \rho^+(v)|_{r\to r^*}. \tag{2.17}$$

2.2. *Realization of* $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ can be realized by using the Drinfeld generators of $U_q(\widehat{\mathfrak{sl}}_N)$ and a Heisenberg algebra. Let $h_i, a^i_m, x^\pm_{i,n}$ $(1 \leqslant i \leqslant N-1 : m \in \mathbb{Z}_{\neq 0}, n \in \mathbb{Z})$, c, d be the standard Drinfeld generators of $U_q(\widehat{\mathfrak{sl}}_N)$ [3, 14]. Their generating functions $x^\pm_i(z), \psi_i(z), \varphi_i(z)$ are called the Drinfeld currents.

$$x^\pm_i(z) = \sum_{n\in\mathbb{Z}} x^\pm_{i,n} z^{-n} \tag{2.18}$$

$$\psi_i\left(q^{\frac{c}{2}}z\right) = q^{h_i} \exp\left((q-q^{-1})\sum_{m>0} a_{i,m} z^{-m}\right) \tag{2.19}$$

$$\varphi_i\left(q^{-\frac{c}{2}}z\right) = q^{-h_i} \exp\left(-(q-q^{-1})\sum_{m>0} a_{i,-m} z^{m}\right) \qquad (1 \leqslant i \leqslant N-1). \tag{2.20}$$

Definition 2.1. *We define 'dressed' currents* $e_i(z,p)$, $f_i(z,p)$, $\psi^\pm_i(z,p)$ $(1 \leqslant i \leqslant N-1)$ *by*

$$e_i(z,p) = u^+_i(z,p) x^+_i(z) \tag{2.21}$$

$$f_i(z,p) = x^-_i(z) u^-_i(z,p) \tag{2.22}$$

$$\psi^+_i(z,p) = u^+_i\left(q^{\frac{c}{2}}z,p\right)\psi_i(z) u^-_i\left(q^{-\frac{c}{2}}z,p\right) \tag{2.23}$$

$$\psi^-_i(z,p) = u^+_i\left(q^{-\frac{c}{2}}z,p\right)\varphi_i(z) u^-_i\left(q^{\frac{c}{2}}z,p\right) \tag{2.24}$$

where

$$u^+_i(z,p) = \exp\left(\sum_{m>0} \frac{1}{[r^*m]_q} a_{i,-m} (q^r z)^m\right) \tag{2.25}$$

$$u^-_i(z,p) = \exp\left(-\sum_{m>0} \frac{1}{[rm]_q} a_{i,m} (q^{-r} z)^{-m}\right). \tag{2.26}$$

Setting $b_{j,m} = \frac{[r^*m]_q}{[rm]_q} a_{j,m}$ (for $m > 0$), $q^{c|m|} a_{j,m}$ (for $m < 0$), we introduce new generators, B^j_m $(1 \leqslant j \leqslant N; m \in \mathbb{Z})$, by

$$-B^j_m + B^{j+1}_m = \frac{m}{[m]_q} b_{j,m} q^{(N-j)m} \qquad \sum_{j=1}^N q^{2jm} B^j_m = 0. \tag{2.27}$$

From this and the commutation relation of the Drinfeld generators $a_{j,m}$, we derive the following commutation relations.

$$\left[B^j_m, B^k_{m'}\right] = m\delta_{m+m',0}\frac{[r^*m]_q[cm]_q}{[rm]_q[m]_q[Nm]_q} \times \begin{cases} [(N-1)m]_q & (j=k) \\ -q^{-mN\mathrm{sgn}(j-k)}[m]_q & (j\neq k) \end{cases} \tag{2.28}$$

for $m, m' \in \mathbb{Z}_{\neq 0}$, $1 \leqslant j, k \leqslant N$. Then defining new currents $k_j(z,p)$ $(1 \leqslant j \leqslant N)$ by

$$k_j(z,p) = :\exp\left(\sum_{m\neq 0}\frac{[m]_q}{m[r^*m]_q}B^j_m z^{-m}\right): \tag{2.29}$$

we obtain the following decomposition.

$$\psi^{\pm}_j\left(q^{\pm(r-\frac{c}{2})}z, p\right) = \kappa q^{\pm h_j}k_j(q^{N-j}z,p)k_{j+1}(q^{N-j}z,p)^{-1}. \tag{2.30}$$

On the other hand, let ϵ_j $(1 \leqslant j \leqslant N)$ be the orthonormal basis in $\mathbb{R}^N$ with the inner product $\langle\epsilon_j, \epsilon_k\rangle = \delta_{j,k}$. Setting $\bar{\epsilon}_j = \epsilon_j - \epsilon$, $\epsilon = \frac{1}{N}\sum_{j=1}^N \epsilon_j$, we have the weight lattice P of type $A^{(1)}_{N-1}$; $P = \oplus_{j=1}^N \mathbb{Z}\bar{\epsilon}_j$. Then, for example, the simple roots α_j $(1 \leqslant j \leqslant N-1)$ of $\mathfrak{sl}_N$ are given by $\alpha_j = -\bar{\epsilon}_j + \bar{\epsilon}_{j+1}$. Let us introduce operators h_α, β $(\alpha, \beta \in P)$ by

$$\left[h_{\bar{\epsilon}_j}, \bar{\epsilon}_k\right] = \langle\bar{\epsilon}_j, \bar{\epsilon}_k\rangle \qquad \left[h_{\bar{\epsilon}_j}, h_{\bar{\epsilon}_k}\right] = 0 = [\bar{\epsilon}_j, \bar{\epsilon}_k] \tag{2.31}$$

$h_\alpha = \sum_j n_j h_{\bar{\epsilon}_j}$ for $\alpha = \sum_j n_j\bar{\epsilon}_j$ and $h_0 = 0$. Note that $\langle\bar{\epsilon}_j, \bar{\epsilon}_k\rangle = \delta_{j,k} - \frac{1}{N}$ and $\left[h_{\alpha_j}, \alpha_k\right] = 2\delta_{j,k} - \delta_{j,k+1} - \delta_{j,k-1} = A_{jk}$. We hence identify $h_{\alpha_j} = -h_{\bar{\epsilon}_j} + h_{\bar{\epsilon}_{j+1}}$ with h_j in the Drinfeld generators of $U_q(\widehat{\mathfrak{sl}}_N)$.

Definition 2.2. *We define the (centrally extended) Heisenberg algebra* $\mathbb{C}\{\hat{\mathcal{H}}\}$ *as an associative algebra generated by* $P_{\bar{\epsilon}_j}$, $Q_{\bar{\epsilon}_j}$ $(1 \leqslant j \leqslant N)$ *and* η_j $(1 \leqslant j \leqslant N-1)$ *with the relations*

$$\left[P_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_k}\right] = \langle\bar{\epsilon}_j, \bar{\epsilon}_k\rangle \qquad \left[P_{\bar{\epsilon}_j}, P_{\bar{\epsilon}_k}\right] = 0 \tag{2.32}$$

$$\left[Q_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_k}\right] = \left(\frac{1}{r} - \frac{1}{r^*}\right)\mathrm{sgn}(j-k)\log q \tag{2.33}$$

$$\left[Q_{\bar{\epsilon}_j}, \eta_k\right] = \frac{1}{r}\mathrm{sgn}(j-k)\log q \tag{2.34}$$

$$[\eta_j, \eta_k] = \frac{1}{r}\mathrm{sgn}(j-k)\log q \tag{2.35}$$

$$\left[P_{\bar{\epsilon}_j}, \eta_k\right] = 0 \qquad \sum_{j=1}^N \eta_j = 0 \tag{2.36}$$

$$[\eta_j, \alpha] = \left[P_{\bar{\epsilon}_j}, U_q(\widehat{\mathfrak{sl}}_N)\right] = \left[Q_{\bar{\epsilon}_j}, U_q(\widehat{\mathfrak{sl}}_N)\right] = [\eta_j, U_q(\widehat{\mathfrak{sl}}_N)] = 0. \tag{2.37}$$

Definition 2.3. *We define the currents* $E_j(v)$, $F_j(v)$, $H^{\pm}_j(v)$ $(1 \leqslant j \leqslant N-1)$ *and* $K_j(v)$ $(1 \leqslant j \leqslant N)$ *by*

$$E_j(v) = e_j(z,p)\,\mathrm{e}^{\bar{\alpha}_j}\,\mathrm{e}^{-Q_{\alpha_j}}(q^{-j+N}z)^{-\frac{P_{\alpha_j}-1}{r^*}} \tag{2.38}$$

$$F_j(v) = f_j(z,p)\,\mathrm{e}^{-\bar{\alpha}_j}(q^{-j+N}z)^{\frac{P_{\alpha_j}-1}{r}}(q^{-j+N}z)^{\frac{h_j}{r}} \tag{2.39}$$

$$H^{\pm}_j(v) = \psi^{\pm}_j(z,p)q^{\mp h_j}\,\mathrm{e}^{-Q_{\alpha_j}}\left(q^{-j+N\pm(r-\frac{c}{2})}z\right)^{(-\frac{1}{r^*}+\frac{1}{r})(P_{\alpha_j}-1)+\frac{1}{r}h_j} \tag{2.40}$$

$$K_j(v) = k_j(z,p)\,\mathrm{e}^{Q_{\bar{\epsilon}_j}}z^{(\frac{1}{r^*}-\frac{1}{r})P_{\bar{\epsilon}_j}}z^{-\frac{1}{r}h_{\bar{\epsilon}_j}+(\frac{1}{r^*}-\frac{1}{r})\frac{N-1}{2N}} \tag{2.41}$$

where $z = q^{2v}$ *and* $\bar{\alpha}_j = -\eta_j + \eta_{j+1}$.

Then it is easy to show that $E_j(v)$, $F_j(v)$, $H_j^{\pm}(v)$ and $K_j(v)$ satisfy the defining relations of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$.

2.3. RLL relation

We next discuss a relation between two elliptic algebras $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. We construct an L-operator by using the half currents and show that it satisfies the dynamical RLL relation which characterizes $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. We use the following abbreviations:

$$P_{j,l} = -P_{\bar{\epsilon}_j} + P_{\bar{\epsilon}_l} = P_{\alpha_j} + P_{\alpha_{j+1}} + \cdots + P_{\alpha_{l-1}} \tag{2.42}$$

$$h_{j,l} = -h_{\bar{\epsilon}_j} + h_{\bar{\epsilon}_l} = h_j + h_{j+1} + \cdots + h_{l-1} \tag{2.43}$$

for $j < l$. From the definition of $\mathbb{C}\{\hat{\mathcal{H}}\}$ and (2.38)–(2.41), we have

$$[K_j(v), P_{k,l}] = (\delta_{j,k} - \delta_{j,l})K_j(v) = [K_j(v), P_{k,l} + h_{k,l}] \tag{2.44}$$

$$[E_j(v), P_{k,l}] = (\delta_{j,k} + \delta_{j+1,l} - \delta_{j,l} - \delta_{j+1,k})E_j(v) \tag{2.45}$$

$$[F_j(v), P_{j,l} + h_{j,l}] = (\delta_{j,k} + \delta_{j+1,l} - \delta_{j,l} - \delta_{j+1,k})F_j(v) \tag{2.46}$$

$$[F_j(v), P_{k,l}] = 0 = [E_j(v), P_{k,l} + h_{k,l}]. \tag{2.47}$$

Definition 2.2. *We define the half currents $F_{j,l}^+(v)$, $E_{l,j}^+(v)$ $(1 \leqslant j < l \leqslant N)$ and $K_j^+(v)$ $(1 \leqslant j \leqslant N)$ by*

$$K_j^+(v) = K_j\left(v + \frac{r+1}{2}\right) \qquad (1 \leqslant j \leqslant N) \tag{2.48}$$

$$\begin{aligned} F_{j,l}^+(v) = a_{j,l} \oint_{C(j,l)} \prod_{m=j}^{l-1} \frac{\mathrm{d}w_m}{2\pi \mathrm{i} w_m} F_{l-1}(v_{l-1}) F_{l-2}(v_{l-2}) \cdots F_j(v_j) \\ \times \frac{\left[v - v_{l-1} + P_{j,l} + h_{j,l} + \frac{l-N}{2} - 1\right][1]}{\left[v - v_{l-1} + \frac{l-N}{2}\right][P_{j,l} + h_{j,l} - 1]} \\ \times \prod_{m=j}^{l-2} \frac{\left[v_{m+1} - v_m + P_{j,m+1} + h_{j,m+1} - \frac{1}{2}\right][1]}{\left[v_{m+1} - v_m + \frac{1}{2}\right][P_{j,m+1} + h_{j,m+1}]} \end{aligned} \tag{2.49}$$

$$\begin{aligned} E_{l,j}^+(v) = a_{j,l}^* \oint_{C^*(j,l)} \prod_{m=j}^{l-1} \frac{\mathrm{d}w_m}{2\pi \mathrm{i} w_m} E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_{l-1}(v_{l-1}) \\ \times \frac{\left[v - v_{l-1} - P_{j,l} + \frac{l-N}{2} + \frac{c}{2} + 1\right]^*[1]^*}{\left[v - v_{l-1} + \frac{l-N}{2} + \frac{c}{2}\right]^*[P_{j,l} - 1]^*} \\ \times \prod_{m=j}^{l-2} \frac{\left[v_{m+1} - v_m - P_{j,m+1} + \frac{1}{2}\right]^*[1]^*}{\left[v_{m+1} - v_m + \frac{1}{2}\right]^*[P_{j,m+1} - 1]^*}. \end{aligned} \tag{2.50}$$

Here $w_m = q^{2v_m}$ and the integration contour $C(j,l)$ and $C^*(j,l)$ are given by

$$\begin{aligned} C(j,l) : |pq^{l-N}z| < |w_{l-1}| < |q^{l-N}z|, \\ |pqw_{m+1}| < |w_m| < |qw_{m+1}| \end{aligned} \tag{2.51}$$

$$C^*(j,l): |p^*q^{l-N+c}z| < |w_{l-1}| < |q^{l-N+c}z|,$$
$$|p^*qw_{m+1}| < |w_m| < |qw_{m+1}| \tag{2.52}$$

where $m = j, j+1, \ldots, l-2$. *The constants* $a_{j,l}$ *and* $a^*_{j,l}$ *are chosen to satisfy*

$$\frac{\kappa a_{j,l} a^*_{j,l}[1]}{q-q^{-1}} = 1. \tag{2.53}$$

2.4. L-operator

Definition 2.3. *By using the half currents, we define the L-operator* $\hat{L}^+(v) \in \mathrm{End}(\mathbb{C}^N) \otimes U_{q,p}(\widehat{\mathfrak{sl}}_N)$ *as* follows:

$$\hat{L}^+(z) = \begin{pmatrix} 1 & F^+_{1,2}(v) & F^+_{1,3}(v) & \cdots & F^+_{1,N}(v) \\ 0 & 1 & F^+_{2,3}(v) & \cdots & F^+_{2,N}(v) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & F^+_{N-1,N}(v) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} K^+_1(v) & 0 & \cdots & 0 \\ 0 & K^+_2(v) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K^+_N(v) \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ E^+_{2,1}(v) & 1 & \ddots & & \vdots \\ E^+_{3,1}(v) & E^+_{3,2}(v) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ E^+_{N,1}(v) & E^+_{N,2}(v) & \cdots & E^+_{N,N-1}(v) & 1 \end{pmatrix}. \tag{2.54}$$

Then a direct comparison with the relations of the half currents leads us to the following conjecture.

Conjecture 2.4. *The L-operator* $\hat{L}^+(v)$ *satisfies the following* $RLL = LLR^*$ *relation:*

$$R^{+(12)}(u_1-u_2, P+h)\hat{L}^{+(1)}(z_1)\hat{L}^{+(2)}(z_2) = \hat{L}^{+(2)}(z_2)\hat{L}^{+(1)}(z_1)R^{+*(12)}(u_1-u_2, P). \tag{2.55}$$

Here $z_i = q^{2u_i}$ $(i = 1, 2)$. *The R-matrix* $R^+(v, P)$ *is the image of the universal R-matrix* $\mathcal{R}(r, \{s_j\})$ *of* $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ *in the evaluation representation* $(\pi_{V_z} \otimes \pi_{V_1})$, $V \cong \mathbb{C}^N$, *given by*

$$R^+(v,s) = \rho^+(v)\bar{R}(v,s) \tag{2.56}$$

$$\bar{R}(v,s) = \sum_{j=1}^N E_{jj}\otimes E_{jj} + \sum_{1\leqslant j<l\leqslant N}(b(v,s_{j,l})E_{jj}\otimes E_{ll} + \bar{b}(v)E_{ll}\otimes E_{jj})$$
$$+ \sum_{1\leqslant j<l\leqslant N}(c(v,s_{j,l})E_{jl}\otimes E_{lj} + \bar{c}(v,s_{j,l})E_{lj}\otimes E_{jl}) \tag{2.57}$$

where $s_{j,l} = \sum_{m=j}^{l-1} s_j$ $(1 \leqslant j < l \leqslant N)$ *and*

$$b(u,s) = \frac{[s+1][s-1][u]}{[s]^2[u+1]} \qquad \bar{b}(u) = \frac{[u]}{[u+1]} \tag{2.58}$$

$$c(u,s) = \frac{[1][s+u]}{[s][u+1]} \qquad \bar{c}(u,s) = \frac{[1][s-u]}{[s][u+1]}. \tag{2.59}$$

And $R^{+*}(v,s) = R^{+}(v,s)|_{r\to r^*}$. *Up to a gauge transformation,* $R^{+}(v,P)$ *coincides with the Boltzmann weight of the* $\widehat{\mathfrak{sl}}_N$ *RSOS model [6].*

The $c = 1$ case, the statement was proved by using the free-field realization [3].

Now let us define the modified L-operator $L^{+}(v,P)$ by

$$L+(z,P) = \hat{L}+(z)\begin{pmatrix} \mathrm{e}^{-Q_{\bar{\epsilon}_1}} & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{-Q_{\bar{\epsilon}_2}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{e}^{-Q_{\bar{\epsilon}_N}} \end{pmatrix} = \hat{L}^{+}(z)\exp\left\{\sum_{m=1}^{N} h^{(1)}_{\epsilon_m} Q_{\bar{\epsilon}_m}\right\}. \qquad (2.60)$$

Here $h^{(1)}_{\epsilon_j} = h_{\epsilon_j}\otimes 1$, $h_{\epsilon_m} \equiv -E_{mm}$ (an $N\times N$ matrix unit). We then show that the modified L-operator depends on neither $Q_{\bar{\epsilon}_j}$ nor η_j and satisfies the dynamical RLL relation of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ [4].

Corollary 2.5.

$$\begin{aligned} &R^{+(12)}(v,P+h)L^{+(1)}(z_1,P)L^{+(2)}(z_2,P+h^{(1)}) \\ &\qquad = L^{+(2)}(z_2,P)L^{+(1)}(z_1,P+h^{(2)})R^{+*(12)}(v,P) \end{aligned} \qquad (2.61)$$

where $u = u_1 - u_2$.

Hence, we regard the elliptic currents $E_j(v)$, $F_j(v)$ $(1\leqslant j\leqslant N-1)$ and $K_j(v)$ $(1\leqslant j\leqslant N)$ in $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as the Drinfeld realization of the elliptic algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ tensored by the Heisenberg algebra:

$$U_{q,p}(\widehat{\mathfrak{sl}}_N) = \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)\otimes_{\mathbb{C}\{P_{\bar{\epsilon}_1},P_{\bar{\epsilon}_2},\ldots,P_{\bar{\epsilon}_{N-1}}\}}\mathbb{C}\{\hat{\mathcal{H}}\}. \qquad (2.62)$$

3. Vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

We here summarize a construction of the type-II vertex operator of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and its dual vertex operator.

3.1. Definition

Let us first define an extension of the $U_q(\widehat{\mathfrak{sl}}_N)$ $(\cong \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N))$ modules by

$$\hat{\mathcal{F}} = \bigoplus_{\mu_1,\ldots,\mu_{N-1}\in\mathbb{Z}} \mathcal{F}\otimes \mathrm{e}^{\mu_1 Q_{\bar{\epsilon}_1}+\cdots+\mu_{N-1}Q_{\bar{\epsilon}_{N-1}}}.$$

Let $\Psi^{*}_{W}(z,P)$ be the type-II intertwining operator of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ [4]. We define the type-II vertex operator $\hat{\Psi}^{*}_{W}(z)$ of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as the following extension:

$$\hat{\Psi}^{*}_{W}(z) = \Psi^{*}_{W}(z,P)\exp\left\{\sum_{j=1}^{N} h^{(1)}_{\epsilon_j}Q_{\bar{\epsilon}_j}\right\} \qquad : W_z\otimes\hat{\mathcal{F}}\longrightarrow\hat{\mathcal{F}}'. \qquad (3.1)$$

From the intertwining relation of the $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ intertwining operators, we derive the following relation for the new operator $\hat{\Psi}^{*}_{W}(z)$:

$$\hat{L}^{+(1)}_{V}(z_1)\hat{\Psi}^{*(2)}_{W}(z_2) = \hat{\Psi}^{*(2)}_{W}(z_2)\hat{L}^{+(1)}_{V}(z_1)R^{+*(12)}_{VW}(u_1-u_2,P-h^{(1)}-h^{(2)}). \qquad (3.2)$$

Let us consider the vector representation $V = W\cong\mathbb{C}^N$ of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$. We denote a basis of V by $\{\mathbf{v}_m\}_{m=1}^{N}$. In this representation, the R-matrix $R^{+}_{VV}(v,P)$ is given by $R^{+}(v,P)$ in

(2.56) and the L-operator $\hat{L}^+_V(z)$ by $\hat{L}^+(z)$ in (2.54). We define the components of the vertex operators by

$$\hat{\Psi}^*_V(q^{-c-1}z)(\mathbf{v}_m \otimes \cdot) = \Psi^*_m(z) \tag{3.3}$$

and the matrix elements of the L-operator $\hat{L}^+(z)$ by

$$\hat{L}^+(z)\mathbf{v}_j = \sum_{1\leqslant m\leqslant N} \mathbf{v}_m L^+(z)_{mj}. \tag{3.4}$$

3.2. Free-field realizations

We here construct a free-field realization of the vertex operators fixing $c = 1$. Let α_j be the simple root operator. We make the standard central extension $[\alpha_j, \alpha_k] = \pi \mathrm{i} A_{jk}$ and set $\hat{\alpha}_j = \alpha_j + \bar{\alpha}_j$, where $\bar{\alpha}_j$ is an element of the Heisenberg algebra $\mathbb{C}\{\hat{H}\}$. Then we have

Proposition 3.1. *The currents $E_j(v)$ and $F_j(v)$ given by*

$$E_j(v) = :\exp\left(-\sum_{m\neq 0}\frac{[rm]_q}{m[r^*m]_q}\left(-B^j_m + B^{j+1}_m\right)(q^{N-j}z)^{-m}\right): \mathrm{e}^{\hat{\alpha}_j} z^{h_j} \mathrm{e}^{-Q_{\alpha_j}} (q^{-j+N}z)^{-\frac{P_{\alpha_j}-1}{r^*}} \tag{3.5}$$

$$F_j(v) = :\exp\left(\sum_{m\neq 0}\frac{1}{m}\left(-B^j_m + B^{j+1}_m\right)(q^{N-j}z)^{-m}\right): \mathrm{e}^{-\hat{\alpha}_j} z^{-h_j} (q^{-j+N}z)^{\frac{P_{\alpha_j}-1}{r}+\frac{h_j}{r}} \tag{3.6}$$

together with $H^{\pm}_j(v)$, $K_j(v)$ given in (2.40)–(2.41) satisfy the commutation relations in definition 2.1 for level $c = 1$.

Now using this free-field realization in (2.48)–(2.50), we obtain a realization of the L-operator $\hat{L}^+(v)$ for $c = 1$. Using this in the 'intertwining relation' (3.2), we can solve it for the type-II vertex operator.

Theorem 3.2. *The highest components of the type-II vertex operator $\Psi^*_N(z)$ are realized in terms of a free field by*

$$\Psi^*_N(z) = :\exp\left(\sum_{m\neq 0}\frac{[rm]}{m[r^*m]}B^N_m z^{-m}\right): \mathrm{e}^{-\bar{\Lambda}_{N-1}} z^{-h_{\bar{\epsilon}_N}} \mathrm{e}^{Q_{\bar{\epsilon}_N}} z^{\frac{1}{r^*}P_{\bar{\epsilon}_N}} z^{(1+\frac{1}{r^*})\frac{N-1}{2N}} \tag{3.7}$$

where $\bar{\Lambda}_{N-1} = \frac{1}{N}(\hat{\alpha}_1 + 2\hat{\alpha}_2 + \cdots + (N-1)\hat{\alpha}_{N-1})$. The other components of the type-II vertex $\Psi^*_j(z)$ $(1 \leqslant j \leqslant N)$ are given by

$$\begin{aligned}\Psi^*_j(z) = a^*_{j,N}\oint_{C^*}\prod_{m=j}^{N-1}\frac{\mathrm{d}w_m}{2\pi\mathrm{i}w_m}\Psi^*_N(v)E_{N-1}(v_{N-1})\cdots E_j(v_j)\\ \times\prod_{m=j}^{N-1}\frac{\left[v_{m+1}-v_m-P_{j,m+1}+\frac{1}{2}\right]^*[1]^*}{\left[v_{m+1}-v_m-\frac{1}{2}\right]^*[P_{j,m+1}-1]^*}.\end{aligned} \tag{3.8}$$

Here $v_N = v$. The integration contour C^* is specified as follows. For the integration contour for w_m $(j \leqslant m \leqslant N-1)$, the poles at $w_m = p^{*n}q^{-1}w_{m+1}$ $(n = 0, 1, \ldots)$ are inside, whereas the poles at $w_m = p^{*-n}qw_{m+1}$ $(n = 0, 1, \ldots)$ are outside.

Theorem 3.3. *The free-field realizations of the type-II vertex operator* $\Psi^*_\mu(z)$ *satisfy the following commutation relation:*

$$\Psi^*_{j_1}(z_1)\Psi^*_{j_2}(z_2) = \sum_{j_1',j_2'=1}^{N} \Psi^*_{j_2'}(z_2)\Psi^*_{j_1'}(z_1) R^{*j_1 j_2}_{j_1' j_2'}(u_1-u_2, P). \tag{3.9}$$

Here we set $R^*(v,P) = \mu^*(v)\bar{R}^*(v,P)$ *with*

$$\mu^*(v) = z^{(\frac{1}{r^*}-1)\frac{N-1}{N}} \frac{\{pq^{2N-2}z\}^*\{q^2z\}^*\{p/z\}^*\{q^{2N}/z\}^*}{\{pz\}^*\{q^{2N}z\}^*\{pq^{2N-2}/z\}^*\{q^2/z\}^*}. \tag{3.10}$$

3.3. The dual vertex operator

The dual of the type-II vertex operator of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ is an operator satisfying

$$\Psi(z) : \hat{\mathcal{F}} \to V_z \otimes \hat{\mathcal{F}}'. \tag{3.11}$$

We define its components in the vector representation by

$$\Psi(z) = \sum_{j=1}^{N} \mathbf{v}_j \otimes \Psi_j(z). \tag{3.12}$$

The following inversion relations hold

$$\Psi_j(z)\Psi^*_k(z') = \delta_{j,k}\frac{g_N z^{1-N}}{1-q^{-N}\frac{z'}{z}} + \cdots$$

$$g_N = \sqrt{-1}^N q^{\frac{N+1}{2r^*}+\frac{N^2-1}{2}} \left(\frac{(p^*q^2;p^*)_\infty}{(p^*;p^*)_\infty}\right)^N \frac{(pq^{2N};q^{2N},p^*)_\infty (q^{2N}q^{-2};q^{2N},p^*)_\infty}{(q^{2N}p^*;q^{2N},p^*)_\infty (q^{2N};q^{2N},p^*)_\infty} \tag{3.13}$$

as $z' \to zq^N$, as well as

$$\sum_{j=1}^{N}\Psi_j(z)\Psi^*_j(z') = \frac{g'_N z^{1-N}}{1-q^N\frac{z'}{z}} + \cdots \qquad \sum_{j=1}^{N}\Psi^*_j(z)\Psi_j(z') = \frac{g'_N z^{1-N}}{1-q^N\frac{z'}{z}} + \cdots \tag{3.14}$$

where

$$g'_N = \sqrt{-1}^{-N} \frac{q^{-\frac{N+1}{2r^*}-\frac{N^2-1}{2}}}{(p^*;p^*)_\infty^{2N-3}(q^{-2};p^*)_\infty^N} \frac{(p;q^{2N},p^*)_\infty (q^{-2};q^{2N},p^*)_\infty}{(p^*;q^{2N},p^*)_\infty (q^{2N};q^{2N},p^*)_\infty}$$

as $z' \to zq^{-N}$. The free-field realizaton is given as follows:

$$\Psi_j(z) = \oint_C \prod_{m=1}^{j-1} \frac{\mathrm{d}w_m}{2\pi \mathrm{i} w_m} \Psi_1(z) E_1(v_1)\cdots E_{j-1}(v_{j-1})$$

$$\times \prod_{m=1}^{j-1} \frac{\left[v_{m-1}-v_m-P_{m-1,j}+\frac{1}{2}\right]^*[1]^*}{\left[v_{m-1}-v_m-\frac{1}{2}\right]^*[P_{m-1,j}-1]^*} \qquad (1 \leqslant j \leqslant N) \tag{3.15}$$

where $v = v_0$ and

$$\Psi_1(z) = :\exp\left(-\sum_{m\neq 0}\frac{[rm]}{m[r^*m]}B^1_m(q^Nz)^{-m}\right): \mathrm{e}^{\bar{\Lambda}_1} z^{h_{\bar{\epsilon}_1}} \mathrm{e}^{-Q_{\bar{\epsilon}_1}} (q^Nz)^{-\frac{1}{r^*}P_{\bar{\epsilon}_1}+\frac{N-1}{2Nr^*}} z^{\frac{N-1}{2N}} \tag{3.16}$$

with $\bar{\Lambda}_1 = \frac{1}{N}((N-1)\hat{\alpha}_1 + (N-2)\hat{\alpha}_2 + \cdots + \hat{\alpha}_{N-1})$. The integration contour C is specified by the condition : for the contour for w_m $(1 \leqslant m \leqslant j-1)$, the poles at $w_m = q^{-1}w_{m-1}p^{*n}$ $(n = 0, 1, 2, \ldots)$ are inside, whereas the poles at $w_m = qw_{m-1}p^{*-n}$ $(n = 0, 1, 2, \ldots)$ are outside.

Remark. The free-field realizations of the vertex operators in theorem 3.2 and of the dual vertex operators are essentially the same as those of the $\widehat{\mathfrak{sl}}_N$ RSOS model obtained in [15, 16].

4. Fusion of the vertex operators

We now consider the fusion of the type-II vertex operator $\Psi_1^*(z_2)$ and its dual $\Psi_1(z_1)$. Namely, we consider a product $\Psi_1(z_1)\Psi_1^*(z_2)$ and investigate the limits to the fusion points $z_1 = q^{-N}p^{*n}z_2$ $(n = 0, 1, 2, \ldots, N)$, where the contour in (3.8) for w_1 gets pinches.

For example, let us consider the case $n = 1$. If we take residues for the poles $w_{N-1} = q^{-1}z_2$, $w_{j-1} = q^{-1}w_j$ $(j = N-1, N-2, \ldots, 3)$, the limit $z_1 \to q^{-N}p^*z_2$ causes pinches in the contour for w_1 at two points $w_1 = q^{-(N-1)}z_2, q^{-(N-1)}p^*z_2$. Similarly, for $1 \leqslant l \leqslant N-2$, if we take residues at the poles $w_{N-1} = q^{-1}z_2$, $w_{j-1} = q^{-1}w_j$ $(j = N-1, N-2, \ldots, N-l+1)$, $w_{N-l} = q^{-1}p^*w_{N-l+1}$, $w_{j-1} = q^{-1}w_j$ $(j = N-l-1, N-l-2, \ldots, 3)$, the same limit $z_1 \to q^{-N}p^*z_2$ causes a pinch in the contour for w_1 at a point $w_1 = q^{-(N-1)}p^*z_2$. Hence in the limit $z_1 \to q^{-N}p^*z_2$, we obtain a total of N terms of contributions from the residues at the N pinching points. Similar consideration leads us to the following results. As $z_1 \to q^{-N}p^{*n}z_2$,

$$\Psi_1(z_1)\Psi_1^*(z_2) = \frac{z_1^{1-N}}{1-q^{-N}p^{*n}\frac{z_2}{z_1}}\Bigg\{C_n\tilde{T}_n(q^{(n-1)r^*}z_2) + \sum_{1\leqslant j_1\leqslant j_2\leqslant\cdots\leqslant j_n\leqslant N}{}' C_{j_1,j_2,\ldots,j_n} :\Lambda_{j_1}(z_2q^{(2n-1)r^*})\Lambda_{j_1}(z_2q^{(2n-3)r^*})\cdots\Lambda_{j_n}(z_2q^{r^*}): \Bigg\} + \cdots. \quad (4.1)$$

Here

$$\tilde{T}_n(z) = \sum_{1\leqslant j_1<j_2<\cdots<j_n\leqslant N} :\Lambda_{j_1}(zq^{(n-1)r^*})\Lambda_{j_2}(zq^{(n-3)r^*})\cdots\Lambda_{j_n}(zq^{-(n-1)r^*}): \quad (4.2)$$

$$\Lambda_j(z) = :\exp\left(\sum_{m\neq 0}\frac{q^{rm}-q^{-rm}}{m}B_m^j z^{-m}\right): q^{-2P_{\bar{\epsilon}_j}}p^{*h_{\bar{\epsilon}_j}}q^{\frac{2(1-N)}{N}}p^{*-\frac{1}{N}-j} \quad (4.3)$$

$$C_n = \sqrt{-1}^N q^{\frac{N+1}{2r^*}+\frac{N^2-1}{2}}\left(\frac{(p^*q^2;p^*)_\infty}{(p^*;p^*)_\infty}\right)^N\left(\frac{1-pq^{-N}}{1-q^{-N}}\right)^n \times\frac{(pq^{2N}p^{*-n};q^{2N},p^*)_\infty(q^{2N-2}p^{*-n};q^{2N},p^*)_\infty}{(q^{2N}p^{*-n};q^{2N},p^*)_\infty(q^{2N}p^{*1-n};q^{2N},p^*)_\infty}. \quad (4.4)$$

In (4.1), $\sum'$ denotes the sum over the complementary set to $1 \leqslant j_1 < j_2 < \cdots < j_n \leqslant N$. $C_{j_1,j_2,\ldots,j_n}$ are constants which are not important here.

The basic operators $\Lambda_j(z)$ $(1 \leqslant j \leqslant N-1)$ coincide with those in the deformed W_N algebra [7, 8]. The expressions for $\tilde{T}_n$ $(1 \leqslant n \leqslant N)$ are almost the same as those of the generating 'currents' of the deformed W_N algebra, but the unit of the q-shift in the arguments in $\Lambda_j(z)$ is different. In an identification of the parameters $p_W = q^{-2}$, $q_W = p = q^{2r}$, where p_W and q_W are p and q in [7, 8], respectively; the unit of the q-shift in [7, 8] is given by p_W, whereas it is $p^* = q^{2(r-1)}$ in our $\tilde{T}_n(z)$. As a consequence, we have

$$\tilde{T}_N(z) =: \Lambda_1(zq^{(N-1)r^*})\Lambda_2(zq^{(N-3)r^*})\cdots\Lambda_N(zq^{-(N-1)r^*}) :\neq 1. \quad (4.5)$$

Therefore, our deformed W algebra generated by $\tilde{T}_n$ $(1 \leqslant n \leqslant N)$ is $\mathfrak{gl}_N$ type instead of $\mathfrak{sl}_N$ type.

On the other hand, since the type-II vertex operator $\Psi^*(z)$ and its dual $\Psi(z)$ are the creation operators of the physical excited particle and anti-particle, it is natural to identify the operators $\tilde{T}_n(z)$ $(1 \leqslant n \leqslant N)$ with the creation operator of their bound states. The S-matrix of the bound state particles are calculated as follows:

$$\tilde{T}_n(z)\tilde{T}_m(w) = S_{n,m}(w/z)\tilde{T}_m(w)\tilde{T}_n(z) \tag{4.6}$$

$$S_{n,m}(z) = \prod_{k=1}^{n}\prod_{l=1}^{m} \varphi_N(zq^{r^*(n-m+2(l-k))}) \tag{4.7}$$

$$\varphi_N(z) = \frac{\Theta_{q^{2N}}(q^2 z)\Theta_{q^{2N}}(p^* z)\Theta_{q^{2N}}(p^{*-1}q^{-2}z)}{\Theta_{q^{2N}}(q^{-2}z)\Theta_{q^{2N}}(p^{*-1}z)\Theta_{q^{2N}}(p^* q^2 z)}. \tag{4.8}$$

Again, this S-matrix is different from the one obtained by Feigin and Frenkel (section 7.2 in [7]) only by the choice of the unit of the q-shift.

The scaling limit of the $\widehat{\mathfrak{sl}}_N$ RSOS model is expected to be the RSOS restriction of the affine Toda field theory with imaginary coupling constant. It is interesting to compare the scaling limit of our S-matrices, $R^*(v, P)$ for the excited particle and $S_{n,m}(z)$ for the bound states, with the bootstrap results [17, 18].

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